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Accuracy Analysis of the Analytical Approximation Model for Electrons Resonant Acceleration in Space Plasma

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Abstract

An electrons resonant acceleration analytical model is investigated. The model describes the process of the strong particle acceleration by the electromagnetic waves packet propagating across the weak constant magnetic field in space plasmas by an analytical formula. The particle strong acceleration is carried out under strict compliance with the Cherenkov resonance conditions. Based on the exact solutions of the second order nonlinear nonstationary differential equation for the wave phase on the charged particle trajectory the initial numerical results are obtained. The calculations using the analytical model are also made. Compering the numerical results, the accuracy of the analytical model was accessed. The model accuracy results are presented in graphical forms. Conclusions on the analytical model applicability of charged particles resonant acceleration studies in space plasma are drawn.

Keywords: Resonant wave-particle interactions, analytical approximation, particle relativistic factor, nonlinear equations, electrons capture, space plasma.

1. Introduction

Investigation of the relativistic particles fluxes generation processes is one of the actual tasks of space plasma physics. It is of considerable interest for the problem of cosmic rays (CR) generation in astrophysics. This is due to the fact that the charged particles surfing on the electromagnetic waves, which was discussed earlier, for example in (Bulanov and Sakharov 1986), (Chernikov et al. 1992), (Dieckmann and Shukla 2006), (Erokhin et al. 1987, 1989, 2007, 2010), (Gribov et al. 1985), (Itin et al. 2000), (Joshi 1984), (Katsouleas and Dawson 1983). This is one of the most effective mechanisms for generating fluxes of ultrarelativistic particles. The correct estimation of the accelerated particles number, their energy spectra, acceleration region characteristic dimensions, require a detailed analysis of charged particles capture conditions in the surfatron acceleration mode. The favorable capture conditions determination of the charged particle and study of the efficiency of particle acceleration under the influence of finite amplitude space localized waves packet is a difficult task. There exists a possibility of multiple charge acceleration by wave packets. In addition, it must be taking into consideration the cyclotron rotation of the particles in an external magnetic field. However, the problem is multiparametric, and to identify patterns of charged particles surfing on the wave packets in space plasma a very large number of numerical calculations needed. In this paper, based on nonlinear numerical calculations, the capture of weakly relativistic charged particles, in surfatron acceleration mode by electromagnetic wave packet, propagating in plasma across a weak external magnetic field H_0 is considered. It should be noted that surfing is possible also for inclined wave propagation, where wave field threshold level is low. Acceleration variant of weakly relativistic particles is discussed, when the period of charges cyclotron rotation is relatively small. For wave amplitude above the threshold on the available time intervals of numerical calculations outside the numerical range favorable for surfing on the waves initial phases on the particle trajectory, charge rotation in the external magnetic field occurs. After several periods of cyclotron rotation (tens, hundreds or more), the condition of Cherenkov resonance occurs and the wave phase on the particle trajectory becomes favorable for the charge capture. Consequently, particle capture by waves packet occurs and ultrarelativistic acceleration begins. Therefore, in the particle pulse area the space capture by the electromagnetic wave turns out to be large enough. These results are helpful for the interpret-tation of experimental data of relativistic particles fluxes in space conditions. Surfing on the electromagnetic waves (Katsouleas and Dawson 1983), (Kichigin 1995, 2001), (Neishtadt et al. 2009), (Shkevov et al. 2013, 2016, 2017), (Sitnov 1988), (Vasiliev et al. 2011), can be a local source (Loznikov and Erokhin 2010) generates fuxes of ultrarelativistic particles in the near space of relatively quiet stars, such as solar heliosphere. In result of this phenomenon, local deviations of the recorded spectrum of CR from the standard power scaling may be observed.

2. Theoretical background

The starting point is the relativistic equations describing the motion of a charged particle, interacting with an electromagnetic wave, with phase velocity, which in the plasma must be less than the speed of light in vacuum. Taking into account the constant of motion, a nonlinear second-order equation for the wave phase on the accelerated charged particle trajectory is obtained. The numerical calculations are carried out for various values of initial parameters.

The surfatron acceleration mechanism is associated with the realization in a magnetoactive plasma of the Cerenkov resonance with the wave-particle interaction, is possible for p-polarized waves with refractive index $N^2 = \varepsilon\perp - (\varepsilon_c^2 / \varepsilon\perp) = 1 - [v (1 - v)] / (1 - u^2 - v)$. An external magnetic field is directed along the z axis $H_0 = H_0 e_z$. Capture in surfing mode occurs for wave amplitudes above the threshold $\sigma = e E / m c$, $\omega > \sigma_c \equiv u \gamma p = u / (1 - \beta p^2) 1 / 2$, $\beta p = \omega / k c$. Consider a wave spectrum with a carrier frequency $\omega_0 = \omega(k_0)$. The parameters are: $u = \omega He / \omega$, $v = (\omega_{pe} / \omega)^2$. Where ω_{He} - is cyclotron frequency of nonrelativistic electrons in plasma, ω_{pe}- is plasma Langmuir frequency $\omega_{pe} = (4 \pi e^2 n_0 / m)^{1/2}$. In perpendicular wave propagation with p-polarization, the wave field's components are E_x, E_y, H_z.

For the phase of the wave packet on the caring frequency $\Psi_0(\tau) = (\omega_0 t - k_0 x)$ can be used nonlinear equation (Erokhin et al. 1987, 1989, 2007, 2010), (Neishtadt et al. 2009), (Shkevov et al. 2013, 2016, 2017), (Sitnov 1988):

$$\gamma\beta_p d^2\Psi_0 / d\tau^2 - (1 - \beta x^2) \cdot (e E_x / m c \omega) - u \beta_y = 0 \qquad (1)$$

Where E_x (x, t) is defined above, $\beta_{p0} = \omega_0 / ck_0$, $\gamma = (1 + h^2 + r_0^2)^{0.5} / (1 - \beta_x^2)^{0.5}$, $r_0 = \gamma\beta_y$ is the initial particle pulse along wave front and the integral of motion $J = \gamma\beta_y + u_0 \beta_{p0} (\Psi_0 - \tau)$ should be taken into account. There exists second integral $\gamma\beta_z = const \equiv h$. Charge velocity component β_x in (1) is given by term $\beta_x = \beta_{p0} [1- (d\Psi_0/d\tau)]$.

3. Task Formulation

In this work accuracy analysis of the electron surfatron acceleration analytical approximation introduced in earlier works is presented. The electron surfatron resonance interactions can be analyticaly expressed by (Erokhin et al. 1987, 1989, 2007, 2010), (Shkevov et al. 2013, 2016, 2017):

$$M (\tau) = \gamma (0) + u\cdot\gamma\cdot\beta (\tau - \tau_c) \qquad (2)$$

Where γ (0) is the particle relativistic factor at the $\tau = 0$, u is defined above, $\beta_p = \beta$ (define by Cherenkov resonance) is wave packet phase velocity, τ_c – moment of the particle capturing; τ_{out} – moment when the particle escapes from the potential well, end of the acceleration period.

Additionally in the Table 3 the following notations is used: $M(\tau)\gamma_{Aini}$ – is the initial value of $\gamma(\tau)$ at the τ_c calculated by analytical approximation; $M(\tau)\gamma_{Aout}$ – is the value of $\gamma(\tau)$ at the τ_{out} calculated by analytical approximation method; DEENS γ_{Eini}– is the initial value of $\gamma(\tau)$ at the τ_c calculated by Differential Equation Exact Numerical Solution method; DEENS γ_{Eout} – is the initial value of $\gamma(\tau)$ at the τ_{out} calculated by Differential Equation Exact Numerical Solution method.

The analysis is carried out by parallel collating of the computing results, both on the proposed model and the numerical solution of second order nonlinear nonstationary differential equations for the wave phase on the charged particle's trajectory. The parallel calculations are made for the 15 cases of different initial phases in range of $\delta\Psi(0) = -3.1 \div 3.1$.

4. Calculation results

The calculations results with initial parameters, shown in the Table 1, are presented in the Table 2 for 70 different cases. Analysis of the Table 2 shows that in two cases no particle trapping (marked with > 70k), in thirty five cases electron initially performs cyclotron rotation with next strong acceleration and in thirty three cases charged particle is trapped immediately (marked with 0). The approximate value of total particle capturing probability is 97%. Further analysis of Table 2 showed that there are two cases with no electrons trapping - marked with (> 70k) ≈ 3 % (in grey); in thirty two cases electrons are trapped immediately ≈ 47 % of calculated cases (dark blue) and thirty six cases (light blue) with initial particle cyclotron rotation and next strong surfatron acceleration ≈ 50 %.

In earlier works (Erokhin et al. 1987, 1989, 2007, 2010), (Neishtadt et al. 2009), (Shkevov et al. 2013, 2016, 2017), (Sitnov 1988), was shown that the most effective particle acceleration takes place in the case when particle capture occurs on the left side of the wave packet. From the other hand, the capture probability is higher when the charged particle is situated in the center of the wave packet structure, similar to the problem discussed in current paper.

For the correct estimation of the particle energy growth, it needed to calculate the relativistic factor value before and after surfatron acceleration period. As it was mentioned above, electron initial relativistic factor is $\gamma(0) = 1.076$ from where $E_{INI} \cong 0.55$ MeV resulting electron initial energy is received. For all cases the dynamics of the relativistic factor $\gamma(\tau)$ analysis showed that for $\delta\Psi_0 = 3.1$ particle is accelerated and energy growth is $\gamma(54830) = 2.83.10^3$, $E_{OUT} \cong 1.45$ GeV. The higher efficiency occurs in case with $\delta\Psi_0 = +1.4$; $\gamma(57280) = 2.957.10^4$ from where E_{OUT} max $\cong 1.51$ GeV.

The dynamics of the particle relativistic factors and its analytical approximation for $\delta\Psi_0 = +3.1$ is presented on the Figure 1. In this case the capture hapens at $\tau_c = 0$ and charged particle is accelerated up to the living the potential well at $\tau_d = 54830$. For this acceleration period the electron reaches the value of the relativistic factor γ (54830) = $2.83.10^3$, $E_{OUT} \cong 1.45$ GeV and the coincidence between proposed function of analytical approximation $M(\tau)$ is accurate up to the third sign - M(54830) = 2830 at the acceleration period.

In this paper our calculations were performed for the following set of initial parameters of the charged particle: $u = 0.20$; $\beta = 0.25$; $h = 0.20$; $g = 0.21$; $a = 0$; $\rho = 70000$, $\sigma = u\,\gamma_p = 1.65\,\sigma_c$; $\Psi_0 = \Psi(0) = 0$ shown in Table 1. Such parameters set corresponds to a low relativistic initial energy of the electron with initial relativistic factor $\gamma(0) = 1.08$ and the resulting electron initial energy is estimated as $E_{INI} \cong 0.55$ MeV. The calculations results are presented in Table 2, where the number written in the cell indicates the time period. The $\delta\Psi_0$ is the value of the wave packet phase and the first point ($\delta\Psi_0 = \Psi(0) = 0$) corresponds to the particle central position in the wave packet, as it was mentioned above. The wave packet electric field amplitude must be above σ_c - $\sigma \geq \sigma_c$, because the surfatron acceleration below the line $\sigma < \sigma_c$ is impossible (Katsouleas and Dawson 1983).

Table 1. *Initial parameters set for the second order nonlinear nonstationary differential equation (1) for the wave phase on the particle trajectory used in the calculations performed. Presented set corresponds to the low relativistic electron energy.*

τ_{max}	φ	u	β_p	h	$g_y(0)$	a	ρ	σ/σc	τ_c	τ_d
70k	0	0.20	0.25	0.20	0.21	0	70000	1.65	0 ÷ 33662	> 50k ÷ 70k

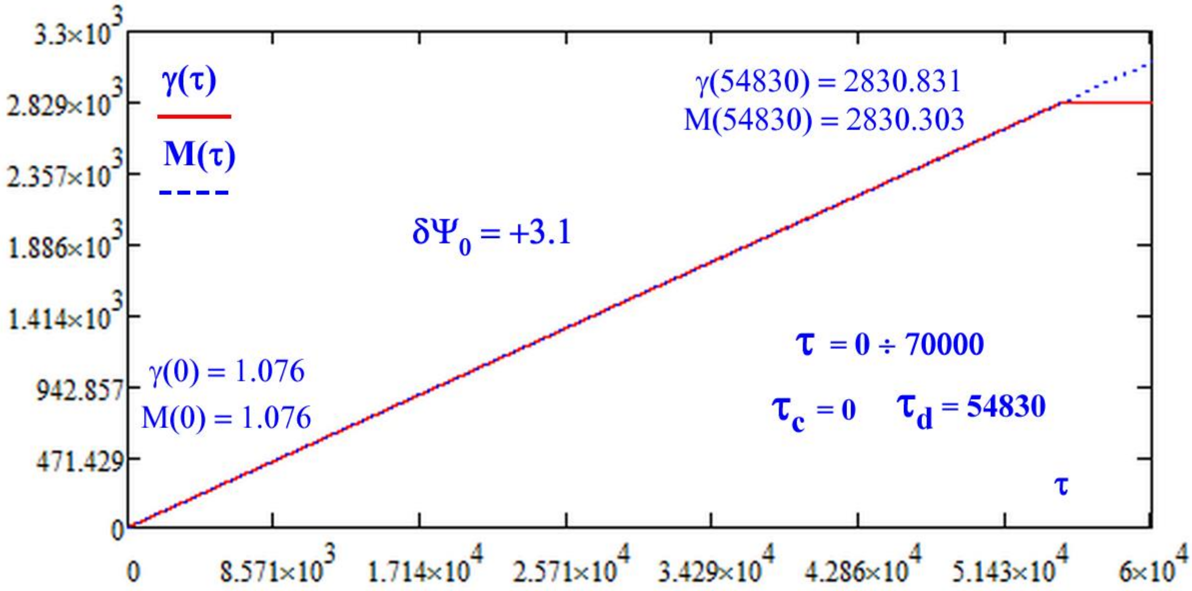

Figure 1. *Plot of charged particle relativistic factor $\gamma(\tau)$ and its analytical approximation $M(\tau)$ for $\delta\Psi(0) = +3.1$ in the range of $\tau = 0 \div 70\,000$ where $\tau_c = 0$ and $\tau_d = 54\,800$.*

Table 2. *The calculations results of 70 cases for the initial phases $\Psi(0) = \phi - \delta\Psi(0)$, where $\phi = 0$ are presented. There exist 68 cases of the particle capturing and strong surfatron acceleration. Capture times τ_c are pointed below the phases values. The green cells correspond to the cases were particle acceleration in surfatron mode takes place. The probability for the surfatron acceleration in this case is about ≈ 97%.*

$\delta\Psi_0$	-3.1	-3.0	-2.9	-2.8	-2.7	-2.6	-2.5	-2.4	-2.3	-2.2
τ_c	0	0	0	19741	15732	6181	9417	27895	3535	16573
$\delta\Psi_0$	-2.1	-2.0	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2
τ_c	3003	11578	17924	33662	3221	10447	13343	27706	19198	3403
$\delta\Psi_0$	-1.1	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2
τ_c	5011	8693	20675	20212	21529	3653	11614	> 70k	23938	10238
$\delta\Psi_0$	-0.1	-0.05	-0.02	-0.01	0	0.01	0.02	0.03	0.05	0.1
τ_c	1058	2836	10684	25605	3451	21183	1491	> 70k	20636	12235
$\delta\Psi_0$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
τ_c	0	0	0	0	0	0	0	0	0	0
$\delta\Psi_0$	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1
τ_c	0	0	0	0	0	0	0	0	0	0
$\delta\Psi_0$	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0	3.1
τ_c	0	0	0	0	0	0	0	0	0	0

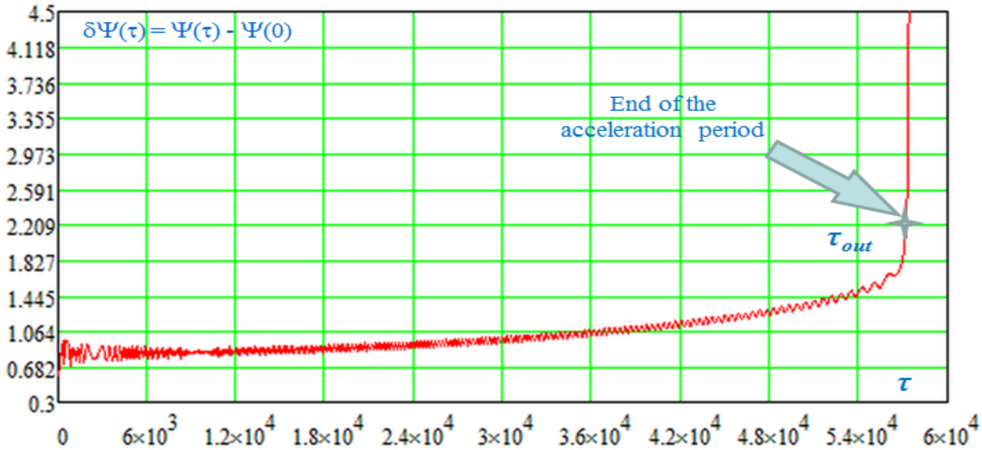

Figure 2. *Acceleration period in the plane of $\Phi(\tau) = d\Psi / d\tau$ and τ is presented. The end of this period is marked as τ_{out}. In this moment particle escape from the potential well and becomes free from the resonance interactions and returns to cyclotron rotation.*

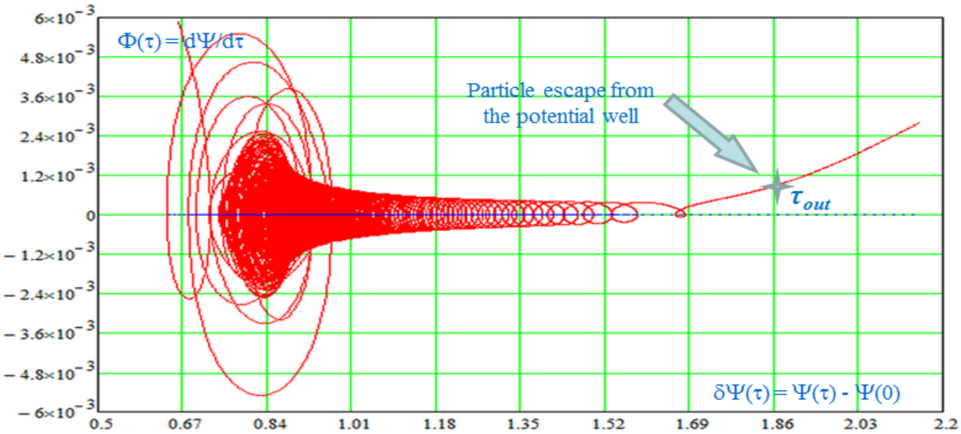

Figure 3. *Phase plane of accelerated particle $\Phi(\tau) = d\Psi / d\tau$ vs $\delta\Psi(\tau) = \Psi(\tau) - \Psi(0)$ in time interval $\tau = 60 \div 5.7\ 10^4$. The end of acceleration period is marked as τ_{out}. The structure of the phase plane resembles the nonlinear oscillators. This kind of trajectories and similar ones proposed in (Shkevov et al. 2017) is called surfatron resonant trajectory.*

Table 3. Part one. *The results of parallel collating of the computing of $\gamma(\tau)$ for the tasks of exact equation solutions and analytic approximation ones. The calculations are carried out for the 15 cases of different initial phases of $\delta\Psi(0) = -3.1 \div 3.1$. The columns ε and ε^2 present the differences in particle relativistic factor $\gamma(\tau)$ by simultainious calculations using two methods.*

$\delta\Psi_0$	τ_c	τ_{out}	M $(\tau)\gamma_{Aini}$	M (τ)	DEENS γ_{Eini}	DEENS γ_{Eout}	ε	ε^2
3.1	0	54830	1.0756	2830.303	1.0756	2830.831	0.528	0.279
2.5	0	56210	1.0756	2901.512	1.0756	2901.942	0.431	0.186
2.0	0	56850	1.0756	2934.536	1.0749	2935.617	1.081	1.169
1.5	0	57000	1.0756	2942.276	1.0745	2943.307	1.032	1.065
1.0	0	56930	1.0747	2938.664	1.0745	2939.286	0.622	0.387
0.5	0	55150	1.0726	2846.813	1.0726	2847.264	0.452	0.204

Table 3. Part two.

$\delta\Psi_0$	τ_c	τ_{out}	M $(\tau)\gamma_{Aini}$	M (τ)	DEENS γ_{Eini}	DEENS γ_{Eout}	ε	ε^2

0.1	12235	66250	1.0608	2788.235	1.0608	2788.823	0.589	0.347
0	3451	57850	2.1637	2809.152	2.1637	2808.777	-0.38	0.141
- 0.1	1058	55220	1.0576	2795.817	1.0576	2796.492	0.675	0.456
- 0.5	11614	65250	4.5475	2772.165	4.5475	2769.513	-2.65	7.033
- 1.0	8693	63550	1.4961	2832.117	1.4961	2832.298	0.181	0.033
- 1.5	13343	66300	6.9934	2742.155	6.9934	2736.672	-5.15	26.481
- 2.0	11578	65500	2.1167	2784.492	2.1167	2784.346	-0.38	0.147
- 2.5	9417	63370	1.4888	2785.464	1.4888	2785.808	0.345	0.119
- 3.1	0	54580	1.078	2818.284	1.078	2817.406	0.878	0.771

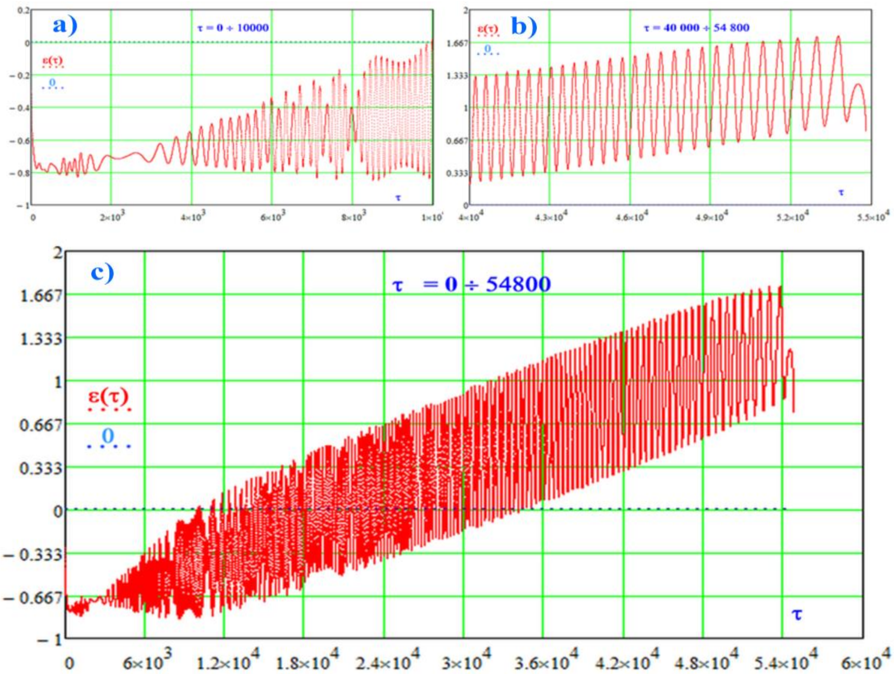

Figure 4. *Function of the absolute difference ε(τ) = γ(τ) - M(τ) at the range of: a) τ = 0 ÷ 10k - the beginning of the acceleration stage; b) τ = 40k ÷ 54.8k - end of the acceleration stage; c) τ = 0 ÷ 54.8k - full length of the acceleration stage for δΨ(0) = 3.1. The maximum in difference value between two calculations methods for τ = 54 800 is 0.018 %.*

5. Conclusions

- Resonant wave-particle interactions of the electrons and electromagnetic wave packet with Lorenz envelope is studied numericaly.
- 70 calculation tasks based on 70 different values for the wave packet phase around Ψ(0) = 0 in range from - 3.1 to +3.1, with maximal calculation length of $\tau_{max} = 7.10^4$ are formed.
- The results presented in Table 2 show cumulative particle capturing probability estimated to approximately ≈ 97%.
- In ≈ 47% of cases, the wave packet traps particle immediately $\delta\Psi_0 = 0$ and in about ≈ 50% electron is trapped after periods with different length of cyclotron rotation.
- The differences in the parallel collating of the computing for the tasks of exact equation solutions and analytic approximation are presented in Table 3.

➢ The maximum in difference value between two calculations methods for ε (τ) in % calculated by (DEENS γ_{Eout} – M(τ)γ_{Aout}) / (DEENS γ_{Eout})*100 is in range of 0.0064% (for δΨ(0) = -1.0) and 0.97% (for δΨ(0) = -1.5).

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