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Solar Gravitational Moments: What Are They and What Do They Do? A Short Comprehensive Review.

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Abstract.

Among all the fundamental solar parameters, mass, diameter, surface gravity, temperature, luminosity..., all well inventoried since several years in reference books, multi-gravitational moments are not yet well documented. Several theoretical estimates have been proposed through different approaches, mainly theory of Figure, helioseismology. We will show their own merits. Exact values of multipolar gravitational moments are important as they are at the crossroads of solar physics, solar astrometry, celestial mechanics, and General Relativity. Their temporal variations are still often neglected; they are yet an essential aspect for constraining solar-cycle modeling or solar-evolution theories. They induced planet-planet inclinations in multi-transiting systems gravitating in the neighboring of a star. This paper emphasizes some key issues to understand the role of these parameters.

1. Introduction.

Up-to-now, key solar parameters as found in books devoted to astronomy are: its diameter (or angular size), mass, absolute magnitude, spectral classification, metallicity, rotation, and sometimes some others such as cycle, modes…, but never the gravitational multipolar moments. Why? Essentially due to (i) their faint order of magnitude (10-7 or less) and (ii) because they are notoriously difficult to measure directly. Detecting a so faint value is at the cutting edge of the current available techniques, even through space dedicated missions. However, they are of fundamental interest, being at the crossroad of celestial mechanics, solar physics, solar astrometry and General Relativity. Their accurate determinations, both theoretically and from observations lead to a better knowledge of the internal properties of our star, down to the core. Extending the issue to stars, by their external influences, gravitational multipolar moments constrain planet-planet inclinations in multi-transiting systems gravitating in the neighboring of the rotating body -a forward thinking issue.

2. What are the gravitational moments?

They are the embodiment of the gravity field induced by the mass of the body, remembering that the gravity field is the sum of the gravitational potential, due to the mass of the body and its centrifugal potential, due to its rotation. And so, they give information on how the distribution of mass and velocity rates act inside the body (the Sun) to finally render the *outer visible shape non spherical*. In other words, determining the gravitational moments is the same as to know the exact equilibrium figure of the body in rotation. From this point of view, such a problem was tackled by many physicists since the XVIII century. Let us quote for instance:

- Maclaurin (1742), Clairaut (1743): in the case of a uniform ellipsoid;
- Jacobi (1834): in the case of spheroids. Study of the stability made by Riemann (1860);
- Bruns (1878): introduced the concept of Figure of a body;
- Radau (1885), Darwin (1889), Wavre (1932), Molodensky (1945), Moritz (1990) and many others provided additional refinements, whilst
- Milne (1923), then Chandrasekhar (1965, 1966) addressed the case of a non-uniform density from the surface down to the center.

Thus, the equilibrium surface of a body in rotation is distorted under the non-uniform distribution of mass and the non-uniform velocity rate, and the question is: how to quantify the departures to sphericity, i.e., how to determine the *shape coefficients*, called also *asphericities coefficients.*

3. Short analysis

Any surfaces can be described by their distance $s(r, q)$ to the center of the body, for each co-latitude q, developed in the form of Legendre polynomials $P_2(cosq)$, $P_4(cosq)$, $P_6(cosq)$, (…) in such a way

$$s(r,\theta) = r\left[1 + s_2(r)P_2(cos\theta) + s_4(r)P_4(cos\theta) + s_6(r)P_6(cos\theta) + (...)\right]$$

that where s_2(r), s_4(r), s_6(r), (...) are functions to be determined, according to the velocity w and the density profile $r(r)$.

Some basic definitions:

1. Oblateness.

f = $(R_{eq}\text{-}R_{pol})/R_{eq}$ or **Dr** = $(R_{eq}\text{-}R_{pol})$

Best sphere passing through R_{eq} and R_{pol} $\quad R_\odot = \sqrt[3]{R_{eq}^2 \times R_{pol}}$

- Numerical application: R_{eq} = 695509.9835 km and R_{pol} = 695504.0331 km

So that: $R_\odot$ = 695508.0000 km (astrometric accuracy!, but needed)

$f = 8.56 \times 10^{-6}$

2. Geodetic parameter: $q = (w^2 R_\odot^3)/GM$

q is a small quantity and thus permits expansion in power series. For the Sun, $q \approx 2.05\ 10^{-5}$

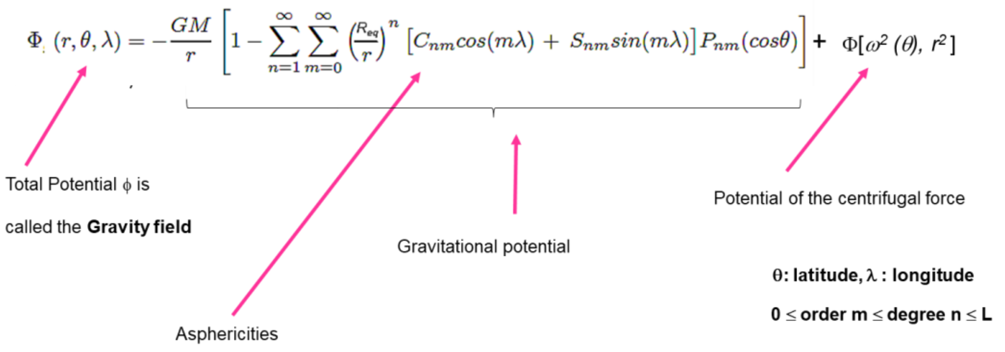
Note that it is straightforward to see (by writing $r(\theta=0°) - r(\theta=90°)$) that Dr (in the general case of a fluid in rotation) is a linear function of the asphericities terms:

$Dr = -(3/2)s_2 - (5/8)s_4 - (21/16)s_6 - (...)s_n$.

General case:

The external gravitational potential $F(r) = - GM/r + (\ldots) + O(1/r^n)$ can be expanded in form of Legendre polynomials:

$F(r, q, l) = - GM/r$ [1 + coefficient * Polynomial expansion $P_{(n,m)}(q)$], where $n = 0, 1, 2, 3 \ldots$, is called the degree and $m = 0, 1, \ldots n$, is called the order of the function under consideration.



The coefficients C_{nm} are called the **tesseral** coefficients.

For the Sun (or stars), for which an axially symmetric distribution of the rotating matter can be assumed (not for the Earth), $S_{nm} = 0, m = 0$, so that

$$\Phi\ (r,\theta,\lambda) = -\frac{GM}{r}\left[1-\sum_{n=2}^{\infty}\left(\frac{R_{eq}}{r}\right)^{n}\left[C_{n0}P_{n0}\left(cos\theta\right)\right]\right]\ \Phi$$

C_{n0} are the **zonal coefficients**.

C_{20} (namely zonal harmonic of degree 2 order 0) is the dynamical flattening of the body, called **quadrupole moment**, more simply denoted by $\boldsymbol{J_2}$, but by convention $C_{20} = - J_2$. In the same way, $C_{40} = - J_4$ is called the octupole moment, $C_{40} = - J_6$ is the hexadecapole moment, and so forth. They have a **physical meaning:** they told us how much the matter deviates from a "perfect" sphere (Figure 1).

The development in power series gives the following results:

$$c_{2,0} = -J_2 = -\left[s_2 + \frac{q}{3} + \frac{11}{7}s_2^2 + \frac{q}{7}s_2 - \frac{223}{196}s_2^3 \right.$$
$$\left. -\frac{61}{196}qs_2^2 - \frac{27}{28}s_4s_2 - \frac{19}{84}qs_4\right]$$

$$c_{4,0} = -J_4 = -\left[s_4 + \frac{36}{35}s_2^2 + \frac{6}{7}qs_2 + \frac{594}{245}s_3^2 + \frac{54}{245}qs_2^2\right]$$

$$c_{6,0} = -J_6 = -\left[s_6 + \frac{120}{77}qs_2^2 + \frac{30}{11}s_4s_2 + \frac{25}{33}qs_4 + \frac{90}{77}s_3^2\right]$$

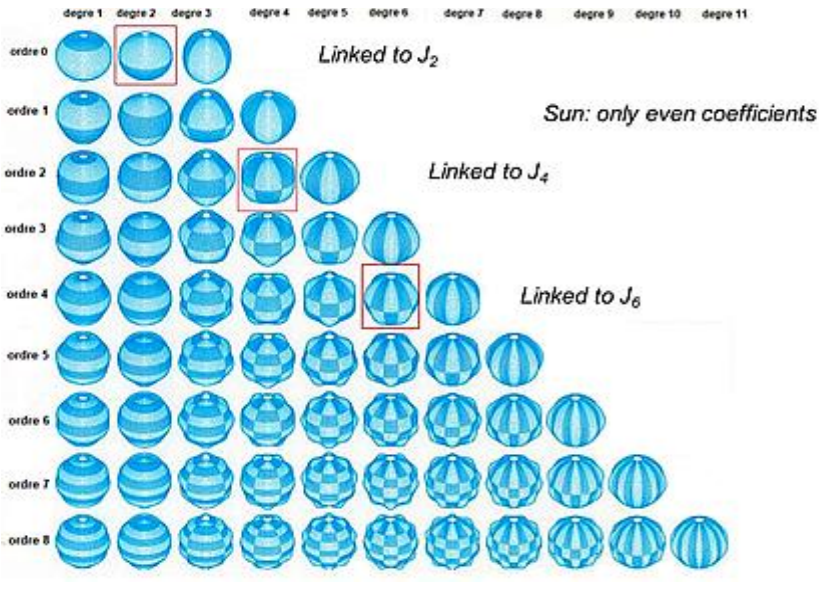


Figure 1. Overview of the Laplace spherical harmonics showing the surface distortions of a rotating fluid body in the general case. The solar case is surrounded by a square. After R. Biancale, 2006 personal communication).

Results:

Taking a solar differential law as found by Javaraiah (2020),

$w = A + B\,sin^2\,(q);\quad A = 14.503\ deg/day;\ B = -2.43\ \ deg/day$, it comes, at $q = 45°$

$J_2 \approx\ 2.54\ 10^{-7},\ J_4 \approx -1.46\ 10^{-10}, J_6 \approx\ 5.31\ 10^{-16}$ (…) .

Note that J_n are of the order of q^n *(q: geodetic parameter).* It is also assumed that q is independent of q. To overcome this difficulty, Dicke (1974) suggested to do computations at the inflexion point of the limb shape curvature. This is justified as it can be seen that the excess of the radius vector over that of the true ellipsoid is obtained at the colatitude for which the change occurs in the external contour shape, i.e. when the radial rotation gradient $\partial\omega/\partial r$ changes in sign. Indeed $\partial\omega/\partial r$ is < 0 at the equator up to $\theta \approx (40 - 55)^{\circ}$ and is positive thereafter[1].

4. **What can be done with J_n?**

4.1. Accurate astrometry

- **Planetary Ephemerides** are developed on the basis of numerical integration of the motion of the planets and the Moon fitted to the most accurate available observations. High precision planetary ephemerides are made independently by three main institutes. (1) The Institute of Applied Astronomy (IAA–RAS) in Saint-Petersburg (Russia), which developed the Ephemerides of Planets and the Moon (EPM), created in the 1970's. (2) The Jet Propulsion Laboratory (JPL, Pasadena, Ca, USA) which created the "Development Ephemerides (DE)" remaining since a long time the basis of the Astronomical Almanac and (3) the "Institut de Mécanique Céleste et de Calcul des Ephémérides" (IMCCE-Paris, F) which developed since the early 80's, analytical solutions for the planetary motion called Intégration Numérique Planétaire de l'Observatoire de Paris (INPOP). As a sub-product of the data fitting of the analytical solutions with observations, is the determination of J_2. Table 1 gives the results, for which the ponderated mean is (2.17 ± 0.6) 10^{-7} on good agreement with the estimation of Pireaux and Rozelot (2003): (2.0 ± 0.4) 10^{-7}.

Name of the ephemeris & Institute	Quadrupole moment (in 10^{-7})	Author & Reference
DE405 JPL (USA)	1.9 ± 0.3	Standish 1998
EPM2004 IAA (Russia)	1.9 ± 0.3	Pitjeva 2005; Pitjeva 2013
INPOP06 IMCCE (F)	1.95 ± 0.55	Fienga 2011
EPM2008 IAA (Russia)	1.92 ± 0.30	Pitjeva 2014a
INPOP08 IMCCE (F)	1.82 ± 0.47	Fienga 2011
DE423 JPL (USA)	1.80	Folkner 2015
INPOP10e IMCCE (F)	1.8 ± 0.25	Verma 2014
EPM2011 IAA (Russia)	2.0 ± 0.2	Pitjeva 2014a
DE430 JPL (USA)	2.1 ± 0.7	Folkner 2015
EPM2013 IAA (Russia)	2.22 ± 0.23	Pitjeva 2014a
INPOP13a IMCCE (F)	2.40 ± 0.20	Verma 2014
INPOP13c IMCCE (F)	2.3 ± 0.25	Fienga 2015
MESSENGER JPL (USA)	2.26 ± 0.09	Park 2017

Table 1: Estimates of the solar quadrupole moment J_2, with their uncertainties, deduced from the Ephemerides. JPL: NASA Jet Propulsion Laboratory in Pasadena (USA). IAA: Institute of Applied Astronomy in Saint Petersburg (Russia). IMCCE: Institut de M´ecanique C´eleste et de Calcul des Eph´em´erides in Paris (F). INPOP13c is an upgraded version of INPOP13a,

[1] The order of magnitude of this outwardgradient is 5.7 10^{16} m^{-1} s^{-1} at the equator.

fitted to LLR (Laser Lunar Ranging) observations, including new observations of Mars and Venus deduced from MEX, Mars Odyssey and VEX tracking data.

4.2. Accurate astrometry

1.2. The perihelion precession of planetary orbits is linked to J_2 and J_4:

$$\dot{\omega}_{J_2} = \frac{3}{2}\frac{nJ_2}{(1-e^2)^2}\left(\frac{R}{a}\right)^2\left(1-\frac{3}{2}\sin^2 i\right)$$

$$\dot{\omega}_{J_4} = -\frac{15}{16}nJ_4\left(\frac{R}{a}\right)^4\left[\frac{3}{(1-e^2)^3}+7\frac{(1+(3/2)e^2)}{(1-e^2)^4}\right]\times\left(\frac{7}{4}\sin^4 i-2\sin^2 i+\frac{2}{5}\right)$$

where $n = \sqrt{GM/a^3}$ is the Keplerian mean motion, R, the Sun's mean equatorial radius, i the angle between the planet's orbit and the Sun's equator.
For instance, Venus' perihelion precession induced by the solar quadrupole mass moment amounts to +0.0026 arcs.y^{-1}, and by the octupole mass moment 4.8 10-9 arcs.y^{-1}.
An analytical development has been made by Xu et al. (2011):

- perihelion precession of Mercury's orbit: $\Delta\omega = 2.95694\times10^5\ J_2$, which is 0.0591 (arcs/yr),
- perihelion precession of Venus's orbit: $\Delta\omega = 6.28801\times10^4\ J_2$, which is 0.01258 (arcs/yr),
- perihelion precession of Mars's orbit: $\Delta\omega = 2.95694\times10^3\ J_2$, which is 0.0013 (arcs/yr).

4.3. Application to stars

Dynamical influence of stellar oblateness may be approximated using the leading order quadrupole terms, neglecting those of order O(f_2). In the case of an exoplanet of mass mp orbiting around its host star of radius r∗ and mass M∗, the disturbing part of the stellar potential is:

$$D = \frac{GM_* m_p}{2a_p}\left(\frac{R_*}{a_p}\right)^2 J_2\left(\frac{3}{2}sin^2 i_p - 1\right)$$

where a_p is the distance of the planet from its host and i_p its inclination orbit.

Ex: Precession rates of planets orbiting the rapidly-rotating main-sequence stars WASP- 33, Kepler-13A and HAT-P-7 reveal associated values of J_2 stars of the order of $\mathbf{10^{-4}}$**.**

5. Conclusion

We emphasized here the need to better determined the solar mutipolar gravitational moments. The solar limb's tiny departure from perfect circularity, i.e. the asphericity, is sensitive to the Sun's otherwise invisible interior conditions2 allowing us to learn empirically about flows and motions there that would otherwise only be guessed from theoretical considerations. There is

[2] Isaak (1999) called this peering inside the Sun from its surface "an open window on the solar interior".

still work to be done, for instance, are the multipolar moments temporal dependent? At last, the solar gravitational moments knowledge can be applied to stars and their planets, highlighting the dynamical influence of stellar oblateness.

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